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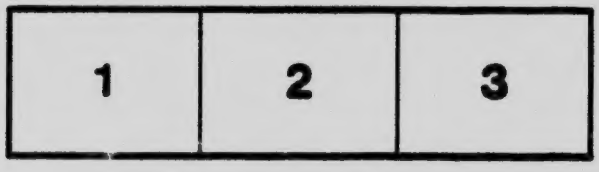

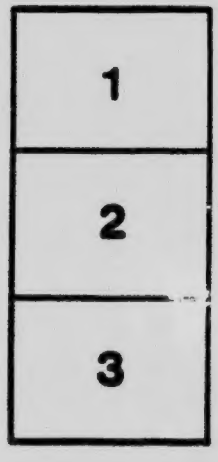

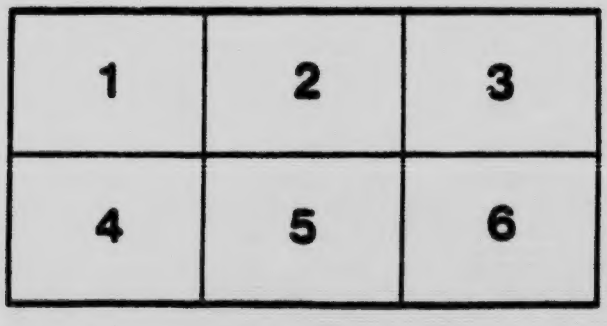

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Abacus of the Altitude and Azimuth of the Pole Star

By E. DEVILLE, LL.D.

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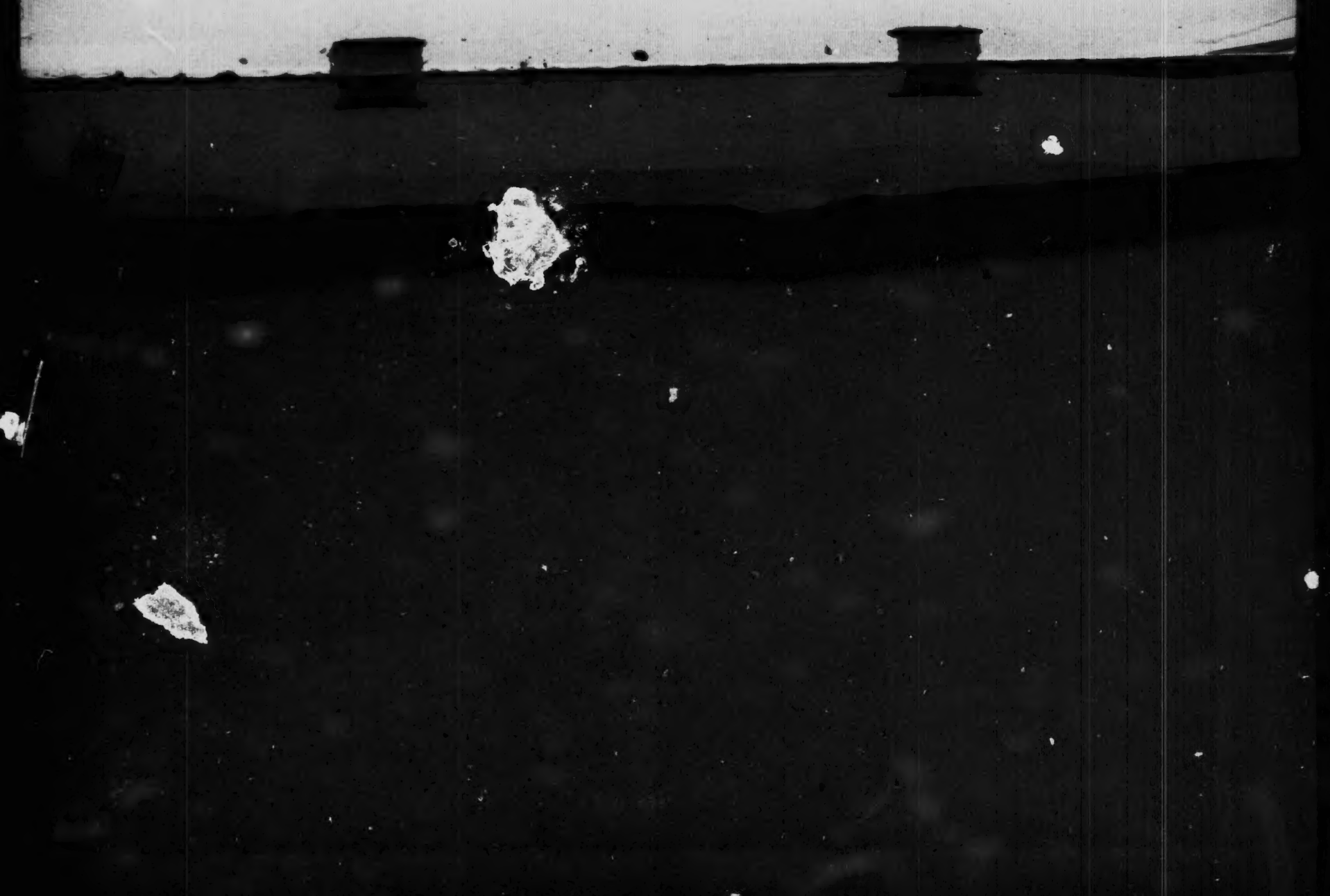

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I.—*Abacus of the Altitude and Azimuth of the Pole Star.*

By E. DEVILLE, LL.D.

(Read May 23rd, 1906.) OCT 29 1923

The boundaries of sections in the land surveys of the Dominion being north and south or east and west lines, it is essential that surveyors who subdivide townships should ascertain frequently the direction of the astronomical meridian, so that they may know the exact bearings of the lines which they are running. The method prescribed for this determination is the observation of the Pole Star in day light. The Star is readily seen an hour after sunrise or before sunset with the telescope of 1½ in. aperture supplied to the Dominion Land Surveyors, provided it is adjusted to bring the Star approximately in the centre of the field. The direction in azimuth is given from the survey lines or by means of the magnetic needle, after which the telescope is set to the altitude of the Pole Star. Sidereal time is given by a common watch, regulated to gain 3 m. 56 s. per day; its error is ascertained from time to time by meridian transits of the sun or stars. To facilitate matters, astronomical field tables are supplied to surveyors. Among other data the tables give the bearing of the Pole Star for every ten minutes and for townships 0, 20, 40, 60 and 80: the bearing at any other time and for any other township is obtained by interpolation. The distance of the Star above or below the pole is also given for calculating the altitude. Although the interpolation for the bearing and the calculation of the altitude are very simple, some surveyors prefer to have no calculation whatever: this condition is fulfilled by the abacus.

Graphic Representation of Equations.

Before explaining the theory of this abacus, it is necessary to recall a few of the principles of the graphic representation of equations. An exhaustive investigation of the subject has been made by d'Ocagne:[1] what is needed for our purpose may be briefly summed up as follows:

If, in the equation of a curve:

$$f_1(x, y, \alpha_1) = 0 \tag{1}$$

successive increments are given to the parameter α_1, to each of these increments corresponds a different curve: the equation thus defines a system of curves (α_1).

[1] Traité de Nomographie by Maurice d'Ocagne, Paris—Gauthier—Villars.

In the same way, the equations:

$$f_2(x, y, \alpha_2) = 0 \tag{2}$$

$$f_3(x, y, \alpha_3) = 0 \tag{3}$$

define the systems of curves (α_2) and (α_3). When three of these curves taken respectively in each of the systems intersect in one point, the corresponding values of the variables α_1, α_2, α_3, satisfy the equation:

$$F(\alpha_1, \alpha_2, \alpha_3) = 0$$

resulting from the elimination of x and y between the equations (1), (2) and (3). The value of any one of the variables can thus be obtained by means of the other two. For instance, if we wish in Fig. 1 to

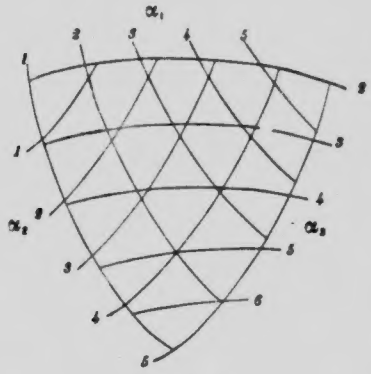

FIG. 1.

find the value of α_3 corresponding to $\alpha_1 = 2$ and $\alpha_2 = 4$, we follow to their intersection the curves marked "2" in the system (α_1) and "4" in the system (α_2): the curve of the system (α_3) passing through this point being marked "5", this number is the required value of α_3.

This kind of abacus is the one most frequently met with, although by no means the best. Usually one of the variables, α_1, is taken as x and another, α_2, as y; α_1 is thus represented by a series of parallels to the y axis, α_2 by a series of parallels to the x axis and α_3 by a series of curves. The use of this abacus requires simultaneous interpolation by estimation between three pairs of lines, an operation not susceptible of much precision. The accuracy may to some extent be increased by drawing more lines, but a limit is soon reached beyond which the number of lines becomes confusing.

To shorten writing, let f_n, ϕ_n, ψ_n, be written instead of $f_n(\alpha_n)$, $\phi_n(\alpha_n)$, $\psi_n(\alpha_n)$, and let us consider the particular case when equations (1), (2), (3), assume the form:

$$\begin{aligned} x f_1 + y \phi_1 + \psi_1 &= 0 \\ x f_2 + y \phi_2 + \psi_2 &= 0 \\ x f_3 + y \phi_3 + \psi_3 &= 0 \end{aligned} \tag{4}$$

Each of these equations defining a system of straight lines, their resultant after the elimination of x and y:

$$\begin{vmatrix} f_1 & \phi_1 & \psi_1 \\ f_2 & \phi_2 & \psi_2 \\ f_3 & \phi_3 & \psi_3 \end{vmatrix} = 0 \tag{5}$$

is represented by three systems of straight lines. Thus an abacus consisting of straight lines only can be constructed whenever the equation to be represented can be put in the form of equation (5).

By the application of the principle of duality, this figure can be transformed into a correlated one such that to straight lines shall correspond points. Each of the equations (4) which, in the first figure,

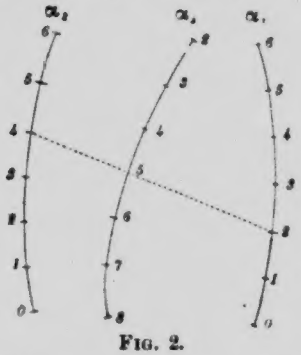

FIG. 2.

defines a system of straight lines tangent to their envelope, defines, in the second figure, points distributed upon a curve, their bearer, as in Fig. 2. Equation (5) which in the first figure means that three straight lines are copunctal, means in the correlated figure that three points are costraight. Instead of following as in Fig. 1 the lines (α_1) and (α_2) to their intersection and finding the line of the system (α_3) which passes through this point, the mode of employment of the new kind of abacus (Fig. 2) consists in joining by a straight line the points (α_1) and (α_2) and reading the graduation at the intersection of the bearer of (α_3). The abacus has gained in simplicity, consisting only of three lines, and the interpolation by estimation instead of being simultaneous between three pairs of lines is now made three times in succession between two divisions of a graduation, a process susceptible of considerable precision.

A convenient way of effecting the transformation is to employ *parallel* instead of *cartesian* co-ordinates. The parallel co-ordinates u and v of a straight line are the distances AM, BN, (Fig. 3) of its inter-

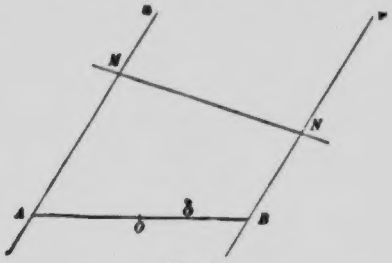

Fig 3.

sections by two parallel lines from the origins A and B selected on these parallels. In this system, an equation of the first degree:

$$au + bv + c = 0 \tag{7}$$

defines a point of which the cartesian co-ordinates may be found as follows: Taking O, centre of AB, as origin, OB as axis of x, a parallel through O to AM and BN as axis of y and designating by δ the distance OB, we have: [1]

$$x = \delta \frac{b - a}{b + a} \tag{8}$$

$$y = \frac{-c}{b + a} \tag{9}$$

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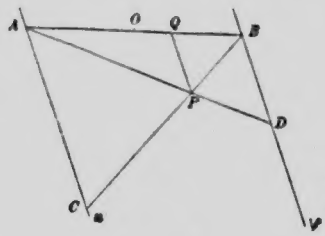
[1] Equation (7) gives for $u = 0$:

$$v = \frac{-c}{b}$$

and for $v = 0$:

$$u = \frac{-c}{a}$$



Taking $AC = \frac{-c}{a}$ and $BD = \frac{-c}{b}$ the point defined by equation (7) is P, intersection of AD and BC, (Fig. 4).

Fig. 4.

Similar triangles give the following proportions:

$$\frac{AQ}{AB} = \frac{QP}{BD}$$

and

$$\frac{BQ}{BA} = \frac{QP}{AC}$$

Abacus of the Azimuth of the Pole Star.

The azimuth of a star in terms of the latitude, hour angle and polar distance, is given by the formula:

$$\tan z = \frac{\tan P \sec L \sin t}{1 - \tan P \tan L \cos t}$$

in which:

z = Azimuth of the star
P = Polar distance of the star
t = Hour angle of the star
L = Latitude.

The azimuth and polar distance of the Pole Star are so small that if the above expression be developed in terms of the powers of z and P, the terms containing powers above the second can, in the case of subdivision surveys, be neglected. Expressing z and P in minutes of arc, we obtain:

$$z \cos L = P \sin t + \frac{P^2}{2} \tan L \sin 2t \sin 1'$$

The surveys of Dominion Lands extend from the 49th parallel of latitude to about township 84, in latitude 56°20′, an interval of 7°20′.

Substituting the values of the different lines, the equations become:

$$\frac{\delta + x}{2\delta} = \frac{y}{\frac{-c}{b}}$$

$$\frac{\delta - x}{2\delta} = \frac{y}{\frac{-c}{a}}$$

hence:

$$\delta + x = 2\delta \frac{by}{-c}$$

$$\delta - x = 2\delta \frac{ay}{-c}$$

Adding up and dividing by 2δ, we have:

$$1 = y\left(\frac{b+a}{-c}\right)$$

or

$$y = \frac{-c}{b+a}$$

Subtracting the second equation from the first one gives:

$$2x = 2\delta y \left(\frac{b-a}{-c}\right)$$

Replacing y by its value and dividing by 2:

$$x = \delta\left(\frac{b-a}{b+a}\right)$$

A mean value of the latitude may therefore be adopted for the last term of the above expression, which is always small.[1] Denoting by L_o this mean value, the equation may be written :

$$\frac{P \sin t + \frac{P^2}{2} \tan L_o \sin 2t \sin 1'}{z} - \cos L = 0$$

now put :

$$P \sin t + \frac{P^2}{2} \tan L_o \sin 2t \sin 1' = \frac{u}{l_1} \tag{10}$$

$$-\cos L = \frac{v}{l_2} \tag{11}$$

and the equation becomes :

$$\frac{u}{z\, l_1} + \frac{v}{l_2} = 0 \tag{12}$$

The value of u is calculated by (10) for hour angle intervals of 10 minutes and laid out on the axis of u, $A\,u$, (Fig. 5), but the sidereal time instead of the hour angle is marked opposite the divisions of the graduation. This time is equal to the sum of the hour angle and right ascension of the Star. The modulus l_1 is the length of one minute of arc on

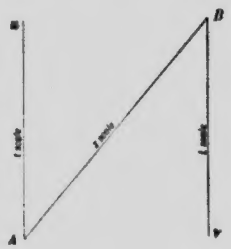


FIG. 5.

$A\,u$; it is selected arbitrarily so as to give suitable proportions to the figure.

In the same way, the values of v or $-\cos L$ are laid out below B on the axis of v, Bv, v being negative. The modulus l_2 is the length of $\cos 0°$; like the modulus l_1, it is selected so as to give suitable propor-

[1] Designating by L_1 and L_2 the extreme values of L, the value of L_o which causes the least maximum error in the azimuth is given by the expression :

$$\tan L_o = \frac{\tan L_1 \cos L_2 + \tan L_2 \cos L_1}{\cos L_1 + \cos L_2}$$

In the present case $L_o = 53°17'$. The error is a maximum for townships 0 and 84 and for hour angles of 3 or 9 hours : it is then equal to 0.″22.

tions to the figure. The number of the township corresponding to the latitude is marked on the divisions of the graduation.

The cartesian co-ordinates of the points defined by (12) are given by (8) and (9):

$$x = \delta \frac{z\,l_1 - l_2}{z\,l_1 + l_2} \tag{13}$$

$$y = o$$

y being equal to zero, the line AB is the bearer of the z scale. The values of x might be calculated from (13) and laid out from the centre of AB, but the graduation can be constructed in a more simple manner. In the first place, we observe that for $z = o$, $x = -\delta$; so the zero of the graduation is at A. For $z = \infty$, $x = \delta$; so the figurative point is at B. Now the scale defined by (13) is a linear scale; therefore it is the image of a regular scale and as its figurative point is at B on the line Bv, it is obtained by laying out a regular scale on a parallel to Bv and projecting it on AB from a projection apex on Bv. This is done as follows:

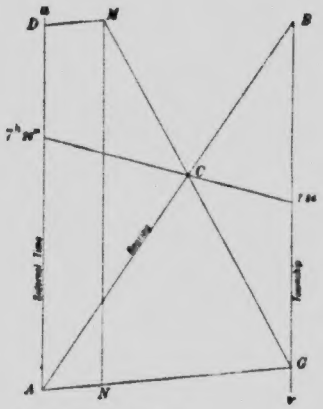


FIG. 6.

Join township 84, (Fig. 6) on the v scale, to 7^h26^m ($t = 6^h$) of the u scale. The intersection C with AB is the end of the useful part of the z scale. The value of z in this case is $\frac{P}{\cos 56°20'}$, let us say 129'.5. With a suitable scale, measure from A on Au a length AD of 129.5. Select on Bv a proper projection apex G; join AG and GC. Through D draw a parallel DM to AG and through the point M where it intersects GC produced, draw MN parallel to Av. The scale used for measuring AD if laid on MN with its zero at N, has its point 129'.5 at M; therefore its projection from G on AC gives the required z scale.

For values of t between 12^h and 24^h, the graduations of both t and z would fall beyond A and increase the size of the figure: this is avoided by changing the sign of u in (10). We have then two graduations for sidereal time on Au, and two graduations for bearing on AB; the second graduations are printed in red to distinguish them. Plate I shows a specimen of the abacus.

Abacus of the Altitude of the Pole Star.

The altitude, h, of a star in terms of the latitude, hour angle and polar distance is given by the formula:

$$\sin h = \sin L \cos P + \cos L \sin P \cos t$$

Let:

$$h = L + x$$

then:

$$\sin L \cos x + \cos L \sin x = \sin L \cos P + \cos L \sin P \cos t$$

P and x are very small. Developing this expression in terms of the powers of P and x, and discarding the terms which contain powers above the second, we find:

$$x = P \cos t - \frac{P^2}{2} \tan L \sin^2 t$$

As before, we adopt a mean value, L_o, for tan L.[1] Allowing 0′.75 for refraction, we may write:

$$H = h + 0'.75$$

Expressing H, L, and P in minutes, we have:

$$L + P \cos t - \frac{P^2}{2} \tan L_o \sin^2 t \sin 1' - H + 0'.75 = 0$$

Putting:

$$(14) \qquad \frac{u}{l_1} = L$$

$$(15) \qquad \frac{v}{l_2} = P \cos t - \frac{P^2}{2} \tan L_o \sin^2 t \sin 1'$$

the equation becomes:

$$(16) \qquad \frac{u}{l_1} + \frac{v}{l_2} - H + 0'.75 = 0$$

The scale of u, (14), is a regular scale of modulus l_1, properly selected, for one minute of latitude. It is laid out on Au, but instead of measuring multiples of l_1, and numbering them in minutes of latitude,

[1] The mean value causing the least maximum error in the altitude is the mean of the extreme values of tan L; it corresponds to $L_o = 52°59'$. The maximum error for $t = 6^h$ or 18^h and for township 0 or 84 is 0′.175.

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we measure multiples of nl_1, n being the number of minutes of latitude in a township, and mark the township number opposite the divisions of the graduation. The scale of v, given by (15), is also a regular scale laid out on Bv with an appropriate modulus l_2; instead of the hour angle, the sidereal time is marked opposite the divisions of the graduation.

The cartesian co-ordinates of the points of the H scale, defined by (16), are given by (8) and (9):

$$x = \delta \frac{l_1 - l_2}{l_1 + l_2}$$

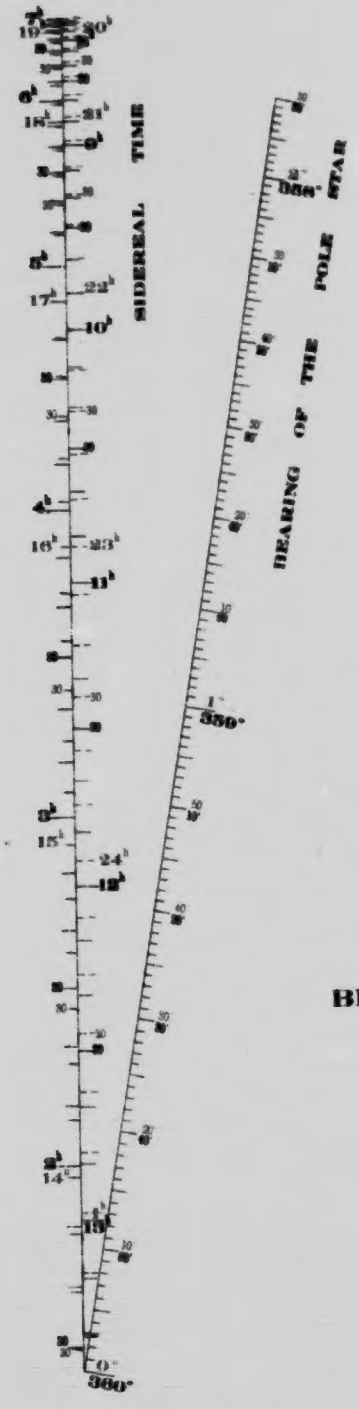
$$y = \left(H - 0'.75\right) \frac{l_1 l_2}{l_1 + l_2}$$

FIG. 7.

x being a constant, the bearer of the H scale is a parallel CD (Fig. 7) to the axes, drawn at a distance x from the centre of AB. The H scale is a regular scale of modulus $\frac{1}{l_1} + \frac{1}{l_2}$, commencing at 0'.75 below the line AB.

The abacus has been made in two parts placed one over the other. The sidereal time scale is identical in both. The divisions of the altitude and township scales have been so arranged that they coincide, but they bear different numbers. The numbers of the second part are printed in red.

Plate II shows a specimen of the abacus.



E. DEVILLE, L.L.D.
Surveyor General

DIAGRAM
OF THE
BEARING OF THE POLE STAR

November and December 1904
September and October 1906
July and August 1907

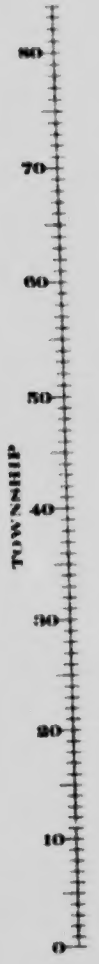


Plate I.

40
30
20
10
0
80
70
60
50

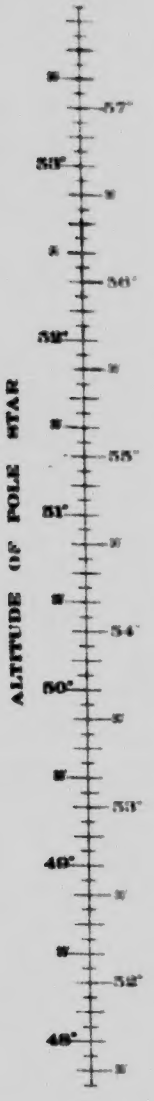


DIAGRAM
OF THE
ALTITUDE
OF THE
POLE STAR

November and December	*1905*
September and October	*1906*
July and August	*1907*

Plate II.